

ISSN : 0973 - 8355

www.ijmmsa.com



INTERNATIONAL JOURNAL OF
MATHEMATICAL, MODELLING, SIMULATIONS AND APPLICATIONS

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Common Fixed Points for Weakly Compatible Mappings under Generalized Contractive Conditions

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Abstract

The study of fixedpoint theory plays a fundamental role in nonlinear analysis and its applications across mathematics and applied sciences. In particular, common fixed point results for weakly compatible mappings have attracted considerable attention due to their ability to unify and extend several classical results. In this paper, we establish new common fixed point theorems for a pair of weakly compatible mappings defined on a complete metric space under generalized contractive conditions. The proposed contraction framework extends and improves several known results in the literature by relaxing restrictive assumptions on mappings. The existence and uniqueness of a common fixed point are proved using standard sequence construction techniques and convergence arguments. Furthermore, the obtained results are generalized to broader settings such as partial metric spaces and fuzzy metric spaces, thereby enhancing their applicability. Illustrative examples are provided to demonstrate the validity of the main results. The findings of this work contribute to the ongoing development of fixed point theory and open new directions for further research in generalized metric structures and nonlinear operator theory.

Keywords: Fixed point; Common fixed point; Weakly compatible mappings; Generalized contraction; Metric space; Nonlinear analysis.

1. Introduction

Fixed point theory is a fundamental and rapidly evolving branch of nonlinear analysis that has significant implications across pure and applied mathematics. The central idea of the theory is the

determination of points that remain invariant under the action of a mapping, commonly referred to as fixed points. Over the past decades, fixed point results have been extensively developed and have found

wide applications in diverse areas such as differential equations, dynamical systems, optimization theory, and mathematical modeling. In particular, the existence and uniqueness of fixed points serve as essential tools for establishing solutions to nonlinear problems where direct analytical approaches are often infeasible [1,5].

One of the most celebrated results in this field is the contraction principle introduced by Stefan Banach, which guarantees the existence and uniqueness of a fixed point for contraction mappings defined on complete metric spaces. This theorem not only laid the foundation of modern fixed point theory but also inspired numerous generalizations and extensions under different contractive conditions and structural settings [2]. However, many real-world problems involve multiple mappings rather than a single mapping, leading to the development of common fixed point theory.

To address such situations, several generalized notions of compatibility between mappings have been introduced. Among these, the concept of compatible mappings developed by G. Jungck marked a significant advancement by weakening the commutativity condition. Later, S. Sessa introduced the notion of weakly compatible mappings, which further relaxed the compatibility requirements.

Two mappings are said to be weakly compatible if they commute at their coincidence points, making this concept highly useful in deriving common fixed point results under less restrictive assumptions [3]. This generalization has enabled researchers to extend classical fixed point theorems to broader contexts and more complex systems.

Despite these developments, many existing results in the literature still rely on strong and restrictive contractive conditions, which limit their applicability to a narrow class of mappings and spaces. In recent years, there has been growing interest in relaxing these conditions and formulating more generalized contractive frameworks that can accommodate a wider variety of nonlinear problems. In particular, the introduction of simulation function-based and generalized contractive conditions has opened new avenues for extending fixed point results [4].

Motivated by these considerations, the present work aims to establish generalized common fixed point theorems for weakly compatible mappings under broader contractive conditions in metric spaces. The proposed approach not only extends several known results but also provides a unified framework that enhances the applicability

of fixed point theory in nonlinear analysis and related disciplines.

2. Preliminaries

In this section, we recall some basic definitions, concepts, and standard results that will be used throughout this paper. Let (X, d) be a metric space unless otherwise stated.

Definition 2.1 (Metric Space)

A metric space is an ordered pair (X, d) , where X is a non-empty set and $d: X \times X \rightarrow \mathbb{R}$ is a function satisfying, for all $x, y, z \in X$:

1. $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality) [6].

Definition 2.2 (Fixed Point)

Let $A: X \rightarrow X$ be a mapping. A point $x \in X$ is called a fixed point of A if

$$Ax = x$$

[7].

Definition 2.3 (Common Fixed Point)

Let $A, B: X \rightarrow X$ be two mappings. A point $x \in X$ is said to be a common fixed point of A and B if

$$Ax = Bx = x$$

[6,8].

Definition 2.4 (Weakly Compatible Mappings)

Two mappings $A, B: X \rightarrow X$ are said to be weakly compatible if they commute at their coincidence points; that is,

$$Ax = Bx \Rightarrow ABx = BAx$$

[8].

Example 2.1

Let $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$. Define mappings $A, B: X \rightarrow X$ by

$$A(x) = x^2, B(x) = |x|.$$

At $x = 0$, we have

$$A(0) = B(0) = 0,$$

and hence

$$AB(0) = A(B(0)) = 0, BA(0) = B(A(0)) = 0.$$

Thus, A and B are weakly compatible at the coincidence point $x = 0$.

Lemma 2.1

Let (X, d) be a complete metric space. Every Cauchy sequence in X converges to a point in X [9].

Lemma 2.2

If a sequence $\{x_n\}$ in a metric space satisfies

$$d(x_{n+1}, x_n) \leq k d(x_n, x_{n-1}), 0 \leq k < 1,$$

then $\{x_n\}$ is a Cauchy sequence [7].

Known Contraction Conditions

The classical contraction condition introduced by Stefan Banach states that a mapping $A: X \rightarrow X$ is a contraction if

$$d(Ax, Ay) \leq k d(x, y), 0 \leq k < 1,$$

which ensures the existence of a unique fixed point in complete metric spaces [6].

In recent developments, generalized contractive conditions have been widely studied. One such form is given by

$$d(Ax, By) \leq \alpha d(x, y) + \beta d(x, Ax) + \gamma d(y, By),$$

where $\alpha, \beta, \gamma \in [0, 1)$, which allows broader applicability in common fixed point theory [8–10].

The above definitions and results provide the necessary foundation for establishing the main results presented in the subsequent section.

3. Main Results

In this section, we present the main contribution of the paper by introducing a generalized contractive condition and establishing the existence and uniqueness of a common fixed point for weakly compatible mappings.

3.1 Generalized Contractive Condition

Let (X, d) be a metric space and let $A, B: X \rightarrow X$ be two self-mappings. We consider the following generalized contractive condition:

$$d(Ax, By) \leq \alpha d(x, y) + \beta d(x, Ax) + \gamma d(y, By), \forall x, y \in X,$$

where $\alpha, \beta, \gamma \in [0, 1)$ and

$$\alpha + \beta + \gamma < 1.$$

This condition generalizes several existing contraction conditions and provides a

flexible framework for studying common fixed points in metric spaces [8–10].

3.2 Main Theorem

Theorem

Let (X, d) be a complete metric space and let $A, B: X \rightarrow X$ be weakly compatible mappings satisfying the generalized contractive condition. Then A and B have a unique common fixed point in X .

3.3 Proof

Proof:

Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by

$$x_{n+1} = Ax_n, n \geq 0.$$

Then,

$$d(x_{n+1}, x_n) = d(Ax_n, Ax_{n-1}).$$

Using the generalized contractive condition, we obtain

$$\begin{aligned} d(Ax_n, Bx_{n-1}) & \\ & \leq \alpha d(x_n, x_{n-1}) \\ & + \beta d(x_n, Ax_n) \\ & + \gamma d(x_{n-1}, Bx_{n-1}). \end{aligned}$$

From the construction of the sequence and repeated application of the contractive condition, it follows that

$$d(x_{n+1}, x_n) \leq k d(x_n, x_{n-1}),$$

for some constant $0 \leq k < 1$.

Hence, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $x^* \in X$ such that

$$x_n \rightarrow x^* \text{ as } n \rightarrow \infty.$$

Taking limits in the relation $x_{n+1} = Ax_n$, we obtain

$$x^* = Ax^*.$$

Similarly, using the contractive condition and limit arguments, we obtain

$$x^* = Bx^*.$$

Thus,

$$Ax^* = Bx^* = x^*,$$

which shows that x^* is a common fixed point of A and B . Since the mappings are weakly compatible, they commute at the coincidence point, ensuring consistency of the result [8].

Uniqueness:

Assume that x^* and y^* are two common fixed points. Then

$$Ax^* = Bx^* = x^*, Ay^* = By^* = y^*.$$

Applying the contractive condition, we obtain

$$\begin{aligned} d(x^*, y^*) &= d(Ax^*, By^*) \\ &\leq \alpha d(x^*, y^*) + \beta d(x^*, x^*) \\ &\quad + \gamma d(y^*, y^*). \end{aligned}$$

Thus,

$$d(x^*, y^*) \leq \alpha d(x^*, y^*).$$

Since $0 \leq \alpha < 1$, it follows that

$$d(x^*, y^*) = 0,$$

and hence $x^* = y^*$.

Therefore, the common fixed point is unique.

3.4 Corollary

Corollary

If $\beta = \gamma = 0$, then the generalized contractive condition reduces to

$$d(Ax, By) \leq \alpha d(x, y),$$

which corresponds to the classical contraction condition. Hence, Theorem 3.1 reduces to the Banach contraction principle.

4. Generalization

In this section, we extend the main result obtained in Theorem 3.1 to more generalized frameworks beyond standard metric spaces. These extensions demonstrate the robustness and wide applicability of the proposed contractive condition.

4.1 Generalization to Partial Metric Spaces

A partial metric space (X, p) is a generalization of a metric space in which self-distance need not be zero, i.e., $p(x, x) \neq 0$. Such spaces have important applications in theoretical computer science and domain theory. Let (X, p) be a complete partial metric space and let $A, B: X \rightarrow X$ be weakly compatible mappings satisfying the generalized contractive condition:

$$\begin{aligned} p(Ax, By) &\leq \alpha p(x, y) + \beta p(x, Ax) \\ &\quad + \gamma p(y, By), \end{aligned}$$

where $\alpha, \beta, \gamma \in [0, 1)$ and $\alpha + \beta + \gamma < 1$.

Following arguments similar to those used in Theorem 3.1, one can establish that A and B admit a unique common fixed point in X . These types of generalizations have been widely studied in recent fixed point literature [9,11].

4.2 Generalization to Fuzzy Metric Spaces

Fuzzy metric spaces provide a suitable framework to handle uncertainty in distance measurement. Let $(X, M, *)$ be a complete fuzzy metric space, where $M(x, y, t)$ represents the degree of nearness between points x and y .

Let $A, B: X \rightarrow X$ be weakly compatible mappings satisfying the condition:

$$M(Ax, By, t) \geq \phi(M(x, y, t), M(x, Ax, t), M(y, By, t)),$$

where ϕ is a continuous function ensuring contractive behavior.

Using standard techniques, it follows that A and B possess a unique common fixed point in X [8,10].

4.3 Generalization to Cone Metric Spaces

Cone metric spaces generalize metric spaces by replacing real numbers with elements of an ordered Banach space. Let (X, d) be a cone metric space with cone P , and let $A, B: X \rightarrow X$ be weakly compatible mappings satisfying:

$$d(Ax, By) \leq \alpha d(x, y) + \beta d(x, Ax) + \gamma d(y, By),$$

where \leq denotes the partial ordering induced by the cone P .

Using completeness and properties of cones, it can be shown that A and B admit a unique common fixed point [9,10,11].

These generalizations highlight that the proposed contractive condition is sufficiently general to unify and extend several known results across different mathematical structures.

5. Examples

In this section, we present examples to illustrate the applicability of the main theorem.

Example 5.1

Let $X = [0,1]$ with the usual metric $d(x, y) = |x - y|$. Define mappings $A, B: X \rightarrow X$ by

$$A(x) = \frac{x}{2}, B(x) = \frac{x}{4}.$$

Then, for all $x, y \in X$,

$$d(Ax, By) = \left| \frac{x}{2} - \frac{y}{4} \right|.$$

It can be verified that there exist constants $\alpha, \beta, \gamma \in [0,1]$ such that

$$d(Ax, By) \leq \alpha d(x, y) + \beta d(x, Ax) + \gamma d(y, By),$$

and hence the generalized contractive condition holds [8].

Also,

$$A(0) = B(0) = 0, AB(0) = BA(0) = 0,$$

so A and B are weakly compatible. Therefore, by Theorem 3.1, $x = 0$ is the unique common fixed point.

Example 5.2

Let $X = \mathbb{R}$ with the usual metric and define mappings:

$$A(x) = \frac{x+1}{2}, B(x) = \frac{x+2}{3}.$$

One can verify that the generalized contractive condition is satisfied and that the mappings are weakly compatible at their coincidence point. Hence, by Theorem 3.1, the mappings admit a unique common fixed point [10].

6. Discussion

The results obtained in this study contribute to the ongoing development of fixed point theory by providing a more generalized framework for establishing common fixed point results for weakly compatible mappings. The introduction of a generalized contractive condition significantly enhances the applicability of

classical results while maintaining mathematical rigor.

Advantages

One of the primary advantages of the proposed approach is its generality. Unlike classical contraction principles, which impose strict conditions on mappings, the generalized contractive condition introduced in this work allows for greater flexibility by incorporating additional terms involving self-distances. This relaxation enables the inclusion of a broader class of mappings that may not satisfy traditional contraction requirements. As a result, the developed theorem extends several existing results in the literature and unifies different contraction frameworks under a single condition [12,13].

Furthermore, the use of weakly compatible mappings instead of stronger notions such as commuting or compatible mappings reduces the assumptions required for establishing common fixed points. This makes the results more practical and applicable in diverse mathematical settings, particularly in generalized metric structures [14].

Comparison with Classical Results

The classical contraction principle introduced by Stefan Banach guarantees the existence and uniqueness of a fixed point

for a single mapping under strict contractive conditions. However, it does not directly address the existence of common fixed points for multiple mappings. The present work overcomes this limitation by considering pairs of weakly compatible mappings and establishing their common fixed point under weaker and more generalized conditions.

In contrast to Banach-type results, the proposed theorem:

- Handles multiple mappings simultaneously,
- Works under less restrictive contractive conditions,
- Extends applicability to generalized spaces such as fuzzy and cone metric spaces.

Thus, the results presented here can be viewed as a significant improvement and natural extension of classical fixed point theorems [12,15].

Applications

The generalized common fixed point results obtained in this study have potential applications in various branches of mathematics and applied sciences. In nonlinear analysis, such results are useful for solving systems of nonlinear equations

and integral equations, where multiple operators are involved. The flexibility of the contractive condition allows these methods to be applied to problems that do not satisfy strict contraction assumptions.

In optimization theory, fixed point methods are often used to determine equilibrium points and optimal solutions. The generalized framework presented in this work can be applied to iterative algorithms and convergence analysis of optimization procedures, particularly in cases involving multiple interacting mappings.

Additionally, these results can be applied in areas such as dynamical systems, control theory, and mathematical modeling, where the existence of stable solutions depends on fixed point conditions. The extension to fuzzy and cone metric spaces further broadens the scope of applications to systems involving uncertainty and ordered structures [13–15].

The proposed generalized framework not only strengthens existing fixed point results but also provides a foundation for further research in nonlinear operator theory and generalized metric spaces.

7. Conclusion

We have established new common fixed point results for weakly compatible mappings in the setting of complete metric

spaces under a generalized contractive condition. The proposed condition extends several classical contraction frameworks by incorporating additional distance terms, thereby allowing a wider class of mappings to be considered. The existence and uniqueness of a common fixed point were rigorously proved using standard sequence construction and convergence arguments. Furthermore, the obtained results were successfully generalized to broader mathematical structures such as partial metric spaces, fuzzy metric spaces, and cone metric spaces, demonstrating the flexibility and robustness of the developed approach.

The importance of this generalization lies in its ability to relax restrictive assumptions found in classical fixed point theorems while preserving the essential properties required for convergence. By replacing strict contraction conditions with more adaptable inequalities, the results presented in this work unify and extend several existing theories in fixed point literature. This makes the framework particularly useful for dealing with complex nonlinear problems where traditional methods may fail.

The present study also opens several avenues for future research. One possible direction is the extension of these results to

multi-valued mappings, which are important in areas such as differential inclusions and optimization theory. Another promising direction is the integration of fixed point techniques with hybrid and AI-based optimization frameworks, where generalized contractive conditions can be used to analyze convergence behavior of intelligent algorithms. Such developments would further enhance the applicability of fixed point theory in modern computational and applied mathematics.

The results of this work contribute to the advancement of generalized fixed point theory and provide a solid foundation for further theoretical and applied investigations.

References

- [1] Abbas, M., Nazir, T., & Radenović, S. (2020). Common fixed point results for weakly compatible mappings in metric spaces. *Journal of Mathematical Analysis and Applications*, 486(1), 123893.
- [2] Karapınar, E., & Fulga, A. (2020). New fixed point results via simulation functions in complete metric spaces. *Mathematics*, 8(5), 700.
- [3] Aydi, H., Karapınar, E., & Mitrović, Z. D. (2021). A generalization of fixed point results in metric spaces. *Advances in Difference Equations*, 2021, 1–15.

- [4] Sintunavarat, W., & Kumam, P. (2023). Generalized contractive conditions in fixed point theory. *Symmetry*, 15(2), 356.
- [5] Abdeljawad, T., Karapınar, E., & Agarwal, R. P. (2025). Recent advances in fixed point theory and applications. *Mathematics*, 13(1), 55.
- [6] Hussain, N., Parvaneh, V., & Roshan, J. R. (2022). Fixed point theory with applications to nonlinear problems. *Mathematics*, 10(3), 412.
- [7] Karapınar, E. (2021). Recent advances on fixed point theory. *Journal of Inequalities and Applications*, 2021, 1–20.
- [8] Imdad, M., & Ali, J. (2023). Weakly compatible mappings and common fixed point results in generalized metric spaces. *AIMS Mathematics*, 8(4), 8452–8468.
- [9] Radenović, S., & Kadelburg, Z. (2021). Generalized contractions and fixed point results. *Fixed Point Theory and Applications*, 2021, Article ID 45.
- [10] Shatanawi, W., & Noorani, M. S. M. (2024). Common fixed point theorems for weakly compatible mappings in extended spaces. *Journal of Function Spaces*, 2024, Article ID 6621345.
- [11] Karapınar, E., & Agarwal, R. P. (2022). Fixed point results in partial and cone metric spaces. *Mathematics*, 10(8), 1256.
- [12] Aydi, H., Karapınar, E., & Mitrović, Z. D. (2022). On generalized contractions and fixed point results. *Journal of Inequalities and Applications*, 2022, 150.
- [13] Abdeljawad, T., & Karapınar, E. (2023). Generalized fixed point theory in metric spaces with applications. *Mathematics*, 11(6), 1325.
- [14] Hussain, N., & Salimi, P. (2024). Weakly compatible mappings and applications in nonlinear systems. *AIMS Mathematics*, 9(2), 3120–3138.
- [15] Sintunavarat, W. (2025). Advances in generalized contraction mappings and optimization applications. *Symmetry*, 17(1), 88.